Power Flow Calculation in Distribution Systems

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Outline

- Conventional power flow calculations in transmission systems
  - Gauss-Seidel method
  - Newton-Raphson method

- Features of electrical distribution networks
  - Ill-conditioned Jacobian matrix in Newton-Raphson method

- Power flow calculations in distribution systems
  - Forward/Backward sweep method
    - Kirchhoff's formulation
      - BIBC & BCBV matrices
    - Dist-flow formulation
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      - Extension to three-phase systems
  - Modified Newton-Raphson method

Power flow calculation

Power flow analysis of power system is used to determine the steady state solution for a given set of bus loading condition.

$$P_{ij} = \sum_{j=1}^{j \in N_i} |V_i||V_j|\left(G_{ij}\cos(\theta_i - \theta_j) + B_{ij}\sin(\theta_i - \theta_j)\right)$$

$$Q_{ij} = \sum_{j=1}^{j \in N_i} |V_i||V_j|\left(G_{ij}\sin(\theta_i - \theta_j) - B_{ij}\cos(\theta_i - \theta_j)\right)$$

Parameters

- Network topology $j \in N_i$
- Line conductance and susceptance G_{ij}, B_{ij}

Variables

- Bus voltage magnitudes and bus phase angles V_i, θ_i
- Line active and reactive power flow P_{ij}, Q_{ij}

Power flow calculation

$$P_{ij} = \sum_{j=1}^{j \in N_i} |V_i||V_j|\left(G_{ij}\cos(\theta_i - \theta_j) + B_{ij}\sin(\theta_i - \theta_j)\right)$$

$$Q_{ij} = \sum_{j=1}^{j \in N_i} |V_i||V_j|\left(G_{ij}\sin(\theta_i - \theta_j) - B_{ij}\cos(\theta_i - \theta_j)\right)$$

Usually, two of the four variables are known for each bus:

- PQ bus (load) at which P and Q are fixed; iteration solves for V and θ .
- PV bus (generator) at which P and V are fixed; iteration solves for θ and Q.
- Slack bus at which the V and θ are fixed; iteration solves for P and Q.

Conventional power flow calculation in transmission systems

Gauss-Seidel method

- It needs to rewrite the equations in an implicit form: $x = h(x)$
- It starts with initial guess: x^0
- Then we update the solution using the following form: $x^{n+1} = h(x^n)$
- It repeats the procedure until converged

It needs to put the equation in the correct form:

$$S_i = V_i I_i^* = V_i \left(\sum_{j=1}^{n} Y_{ij} V_j \right)^* = V_i \sum_{j=1}^{n} Y_{ij}^* V_j^* \qquad S_i^* = V_i^* I_i = V_i^* \sum_{j=1}^{n} Y_{ij} V_j$$

$$\frac{S_i^*}{V_i^*} = \sum_{j=1}^{n} Y_{ij}^* V_j^* = Y_{ii} V_i + \sum_{j=1, j \neq i}^{n} Y_{ij} V_j$$

The update rule for each bus voltage:

$$V_i^{n+1} = \frac{1}{Y_{ii}} \left(\frac{S_i^*}{V_i^{n*}} - \sum_{j=1, j \neq i}^{n} Y_{ij} V_j^n \right) = h_i(V_1^n, V_2^n, \dots, V_I^n)$$

Conventional power flow calculation in transmission systems

Newton-Raphson method

- x is the vector of θ and V for all the buses, except for the slack bus.
- Active and reactive power balance equation: $f(x)$

$$x^{n+1} = x^n - J^{-1}(x^n)f(x^n)$$

where $x = \begin{bmatrix} \theta \\ V \end{bmatrix}$ $f(x) = \begin{bmatrix} P(x) \\ Q(x) \end{bmatrix}$ $\mathrm{J}(x) = \begin{bmatrix} \frac{\partial P}{\partial \theta} & \frac{\partial P}{\partial V} \\ \frac{\partial Q}{\partial \theta} & \frac{\partial Q}{\partial V} \end{bmatrix}$

Compared to Gauss-Seidel method, Newton-Raphson method has a faster convergence rate, but each iteration takes much longer time. Also, Newton-Raphson is more complicated to code.

How to accelerate?

Accelerated (convergence) Gauss-Seidel method

- Previously, it calculates each value x as: $x^{n+1} = h(x^n)$
- To accelerate convergence, the above equation can be rewritten as: $x^{n+1} = x^n + h(x^n) - x^n$
- Acceleration parameter α: $x^{n+1} = x^n + \alpha(h(x^n) - x^n)$
- Larger value of α **may** result in faster convergence

Decoupled Newton-Raphson method

- Approximation of the Jacobian matrix is used to decouple the real and reaction power equations.

Assume $(\theta_i - \theta_j) \approx 0$, thus $\sin(\theta_i - \theta_j) \approx 0$

$$\frac{\partial P_i}{\partial V_j} = |V_i| B_{ij} \sin(\theta_i - \theta_j) \approx 0$$

$$\frac{\partial Q_i}{\partial \theta_j} = -|V_i||V_j| B_{ij} \sin(\theta_i - \theta_j) \approx 0$$

$$\mathrm{J}(x) = \begin{bmatrix} \frac{\partial P}{\partial \theta} & 0 \\ 0 & \frac{\partial Q}{\partial V} \end{bmatrix}$$

$$\theta^{n+1} = \theta^n - \left[\frac{\partial P}{\partial \theta}\right]^{-1} P(x^n)$$

$$V^{n+1} = V^n - \left[\frac{\partial Q}{\partial V}\right]^{-1} Q(x^n)$$

Disadvantages of Gauss-Seidel and Newton-Raphson

Traditional Gauss-Seidel method and Newton-Raphson method need the calculation of the **Y matrix** and **Jacobian matrix**.

When solving large power systems, the most difficult computation task is inverting the **Y matrix** and **Jacobian matrix**:

- Inverting a full matrix needs an order of n^3 operation, meaning the amount of computation increases with the cube of the size n.
- This amount of computation can be decreased substantially by recognizing both Y matrix and Jacobian matrix are sparse matrices.
- Using sparse matrix methods results in a computational order of about $n^{1.5}$.

Features of electrical distribution networks

Because of the following special features in **distribution network**, the Y matrix and Jacobian matrix ceases to be diagonally dominant and convergence problems arise in power flow solutions that rely on its inverse [1].

- Radial or near radial structure
- High R/X rations
- Un-transposed lines
- Multi-phase, unbalanced, grounded or ungrounded operation
- Multi-phase, multi-mode control distribution equipment
- Unbalanced distributed load
- Extremely large number of branches/nodes

Thus, traditional Gauss-Seidel method and Newton-Raphson method have lost popularity due to their poor convergence in distribution system studies.

[1] C. S. Cheng and D. Shirmohammadi, "A three-phase power flow method for real-time distribution system analysis," in *IEEE Transactions on Power Systems*, vol. 10, no. 2, pp. 671-679, May 1995.

Convergence of Newton's method for distribution systems

Condition number defines the condition of a matrix with respect to the computing problem. $\lambda_{max}(J)$ and $\lambda_{min}(J)$ are maximum and minimum eigenvalues of matrix J.

$$k(J) = \frac{\lambda_{max}(J)}{\lambda_{min}(J)}$$

A very high value of the condition number of matrix J indicates that:

- The system is ill-conditioned, the computed values are very sensitive to small changes in input values.
- The matrix J is invertible.

Tab.1 Maximum and Minimum Eigenvalues and Condition Number [2]

Type of System	No. of Buses	Maximum Eigenvalue	Minimum Eigenvalue	Condition Number (k)*	Remarks
Well-conditioned system	30	$.1087\times10^3$	$.2322\times10^0$	$.4681\times10^3$	Moderately well-conditioned (fair k)
Ill-conditioned system	11	$.1222\times10^3$	$.1126\times10^0$	$.1086\times10^4$	Ill-conditioned (bad k)
	13	$.2905\times10^2$	$.1442\times10^{-1}$	$.2014\times10^4$	Ill-conditioned (bad k)
	43	$.2426\times10^4$	$.9476\times10^{-1}$	$.2560\times10^5$	Ill-conditioned (bad k)

* Ideal value of condition number: k = 1

[2] S. C. Tripathy and G. S. S. S. K. Purge Prasad, "Load flow solution for ill-conditioned power systems by quadratically convergent Newton-like method," in *IEE Proceedings C - Generation, Transmission and Distribution*, vol. 127, no. 5, pp. 273-280, September 1980.

Forward/Backward Sweep-based Algorithm

Methods developed for the solution of ill-conditioned radial distribution systems may be divided into two categories [3]:

- Forward and/or backward sweep
  - Kirchhoff's formulation
    - BIBC & BCBV
  - Quadratic equation-based algorithm
    - Dist-Flow
- Modification of existing methods
  - Modified N-R method

Forward/backward sweep-based power flow algorithm generally takes advantage of the radial network topology and consists of forward and backward sweep processes.

- The forward sweep is mainly the node voltage calculation from the sending end to the far end of the lines.
- The backward sweep is primarily the branch current or power summation from the far end to the sending end of the lines.

[3] U. Eminoglu & M. H. Hocaoglu, "Distribution Systems Forward/ Backward Sweep-based Power Flow Algorithms: A Review and Comparison Study', in *Electric Power Components and Systems*, 37:1, 91-110, 2008

Forward/Backward Sweep-based Algorithm

Fig.1 shows a linear ladder network [4].

- For the ladder network, it is assumed that all of the line impedances and load impedances are known along with the voltage (V_S) at the source.
- The solution for this network is to perform the "forward" sweep by calculating the voltage at node 5 (V_5) under a no-load condition.
- With no load currents there are no line currents, so the computed voltage at node 5 will equal that of the specified voltage at the source.
- The "backward" sweep commences by computing the load current at node 5.

The load current I_5 is

$$I_5 = \frac{V_5}{ZL_5}$$

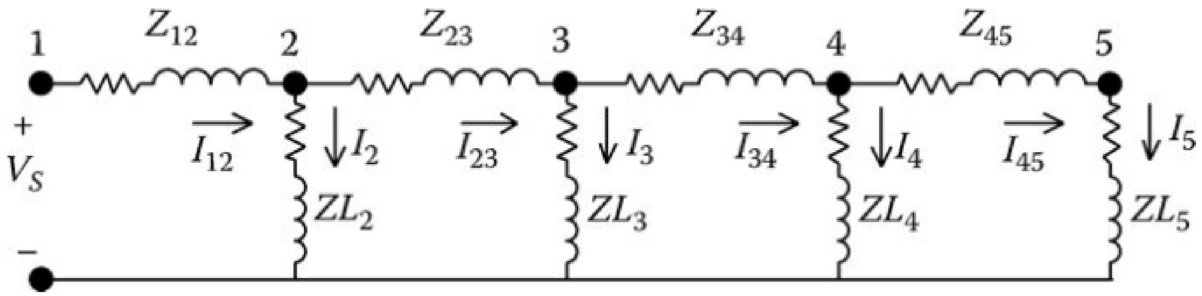


Fig.3 Linear Ladder network [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

For this "end node" case, the line current I_{45} is equal to the load current I_5. The "backward" sweep continues by applying Kirchhoff's voltage law (KVL) to calculate the voltage at node 4:

$$I_{45}=I_5 \qquad V_4 = V_5 + Z_{45} \cdot I_{45}$$

The load current I_4 can be determined and then Kirchhoff's current law (KCL) applied to determine the line current I_{34}:

$$I_4 = \frac{V_4}{ZL_4} \qquad I_{34} = I_{45} + I_4$$

KVL is applied to determine the node voltage V_3. The backward sweep continues until a voltage (V_1) has been computed at the source.

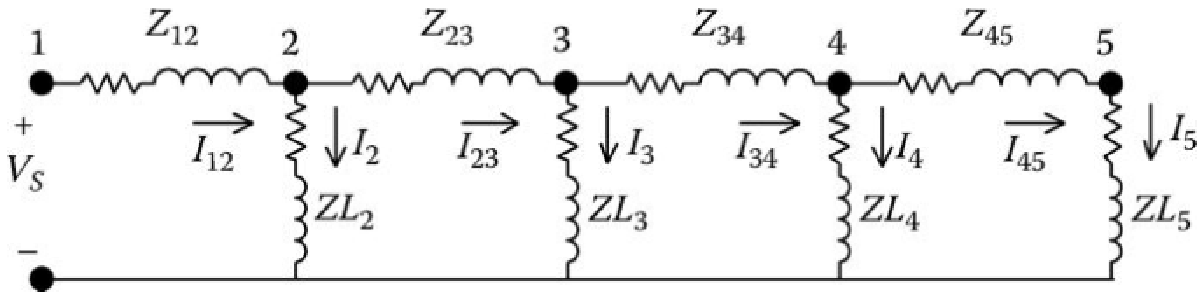
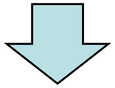


Fig.3 Linear Ladder network [4]

$$V_3 = V_4 + Z_{34} \cdot I_{34}$$

$$\cdots$$



$$V_1$$

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

The computed voltage V_1 is compared to the specified voltage V_S. There will be a difference between these two voltages. The ratio of the specified voltage to the compute voltage can be determined as

$$Ratio = \frac{V_S}{V_1}$$

Since the network is linear, all of the line and load currents and node voltages in the network can be multiplied by the ratio for the final solution to the network.

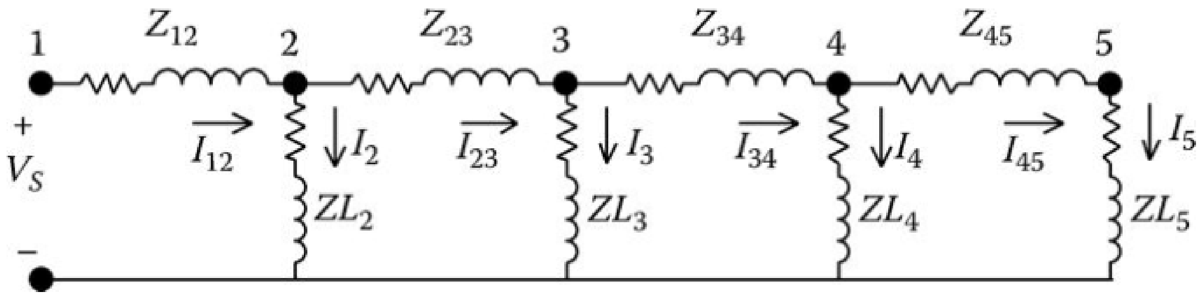


Fig.3 Linear Ladder network [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

The linear network of Fig.3 is modified to a nonlinear network by replacing all of the constant load impedances by constant complex power loads as shown in Fig.4.

As with the linear network, the "forward" sweep computes the voltage at node 5 assuming no load. As before, the node 5 (end node) voltage will equal that of the specified source voltage. In general, the load current at each node is computed by

$$I_n = \left(\frac{S_n}{V_n}\right)^*$$

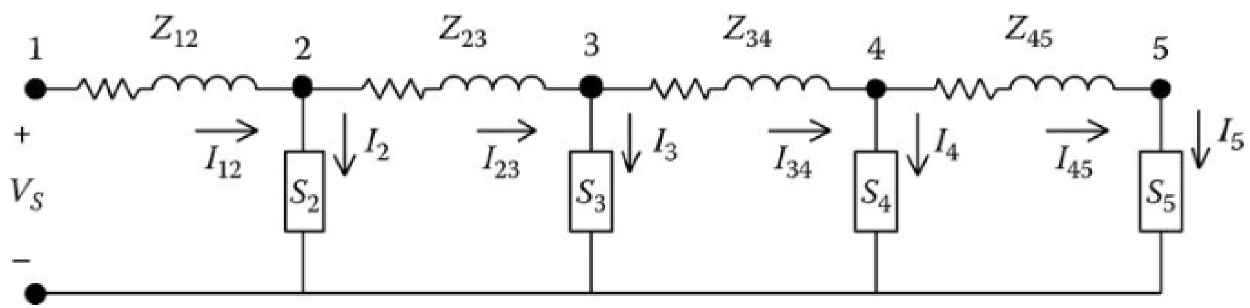


Fig. 4 Nonlinear ladder network [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

The "backward" sweep will determine a computed source voltage V_1.

- As in the linear case, this first "iteration" will produce a voltage that is not equal to the specified source voltage V_S. Because the network is nonlinear, multiplying currents and voltages by the ratio of the specified voltage to the computed voltage will not give the solution.

- The most direct modification using the ladder network theory is to perform a "forward" sweep. The forward sweep commences by using the specified source voltage and the line currents from the previous "backward" sweep. KVL is used to compute the voltage at node 2 by

$$V_2 = V_s - Z_{12} \cdot I_{12}$$

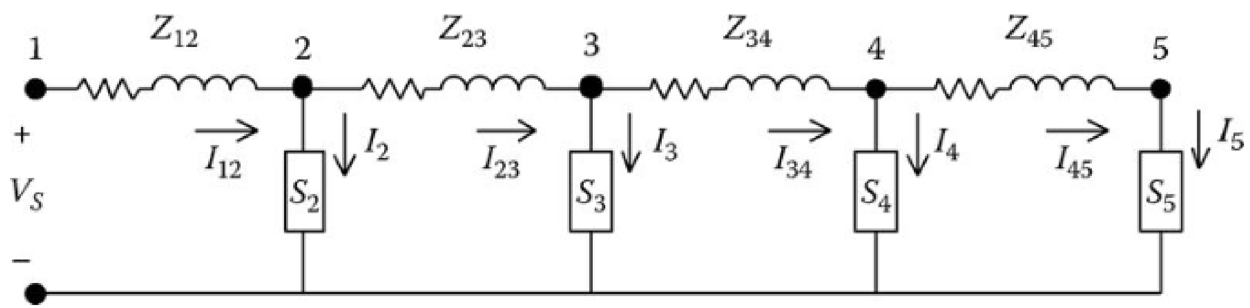


Fig. 4 Nonlinear ladder network [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

This procedure is repeated for each line segment until a "new" voltage is determined at node 5.

- Using the "new" voltage at node 5, a second backward sweep is started that will lead to a "new" computed voltage at the source.
- The procedure shown earlier works but, in general, will require more time to converge. A modified version is to perform the "forward" sweep calculating all of the node voltages using the line currents from the previous "backward" sweep.
- The new "backward" sweep will use the node voltages from the previous "forward" sweep to compute the new load and line currents.
- In general, this modification will require more iterations but less time to converge. In this modified version of the ladder technique, convergence is determined by computing the ratio of difference between the voltages at the $n - 1$ and n iterations and the nominal line-to-neutral voltage. Convergence is achieved when all of the phase voltages at all nodes satisfy

$$\frac{\left||V_n| - |V_{n-1}|\right|}{V_{nominal}} \leq Specified\ tolerance$$

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Example

Use the modified ladder method to compute the load voltage.

A single-phase lateral is shown in Fig.5. The line impedance is

$$z = 0.3 + j0.6\ \Omega/mile$$

The impedance of the line segment 1–2 is

$$Z_{12} = (0.3 + j0.6) \cdot \frac{3000}{5280} = 0.1705 + j0.3409\ \Omega$$

The impedance of the line segment 2–3 is

$$Z_{23} = (0.3 + j0.6) \cdot \frac{4000}{5280} = 0.2273 + j0.4545\ \Omega$$

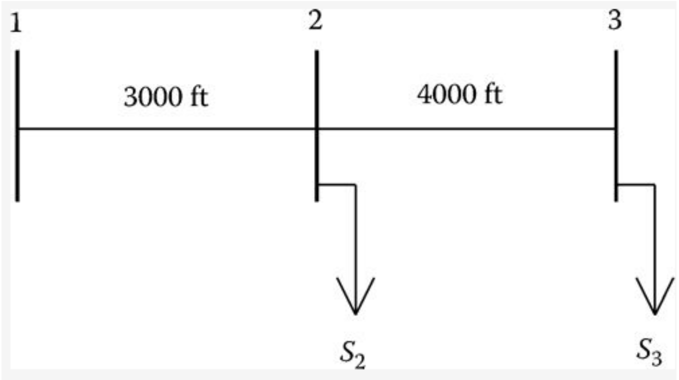


Fig.5 Single-phase lateral [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Example

The loads are

$$S_2 = 1500 + j750\ (kW + jkvar) \qquad S_3 = 900 + j500\ (kW + jkvar)$$

The source voltage at node 1 is 7200 V.

Set initial conditions:

$$I_{12} = I_{23} = 0 \qquad V_{old} = 0 \qquad Tol. = 0.0001$$

***The first forward sweep*:**

$$V_2 = V_s - Z_{12} \cdot I_{12} = 7200\angle 0 \qquad V_3 = V_2 - Z_{23} \cdot I_{23} = 7200\angle 0$$

$$Error = \frac{\left||V_3| - |V_{old}|\right|}{7200}$$
$$= 1\ (greater\ than\ Tol, start\ backward\ sweep)$$

$$V_{old} = V_3$$

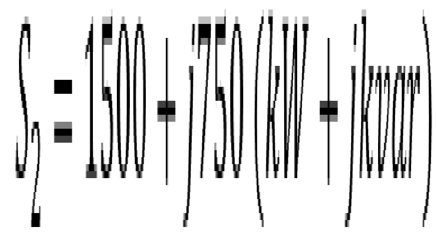


Fig.5 Single-phase lateral [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Example

***The first backward sweep*:**

$$I_3 = \left(\frac{(900 + j500) \cdot 1000}{7200\angle 0}\right)^* = 143.0\angle -29.0\ A$$

The current flowing in the line segment 2–3 is

$$I_{23} = I_3 = 143.0\angle -29.0\ A$$

The load current at node 2 is

$$I_2 = \left(\frac{(1500 + j750) \cdot 1000}{7200\angle 0}\right)^* = 232.9\angle -27.5\ A$$

The current in line segment 1–2 is

$$I_{12} = I_{23} + I_2 = 373.8\angle -27.5\ A$$

***The second forward sweep*:**

$$V_2 = V_2 - Z_{12} \cdot I_{12} = 7084.5\angle -0.7 \qquad V_3 = V_2 - Z_{23} \cdot I_{23} = 7025.1\angle -1.0$$

$$Error = \frac{\left||V_3| - |V_{old}|\right|}{7200} = \frac{|7084.5 - 7200|}{7200} = 0.0243(greater\ than\ Tol, start\ backward\ sweep)$$

$$V_{old} = V_3$$

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Example

At this point, the second backward sweep is used to compute the new line currents. Then it is followed by the third forward sweep.

After four iterations, the voltages have converged to an error of 0.000017 with the final voltages and currents of

$$[V_2]=7081.0\angle-0.68$$

$$[V_3]=7019.3\angle-1.02$$

$$[I_{12}]=383.4\angle-28.33$$

$$[I_{23}]=146.7\angle-30.07$$

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

With reference to Fig.6, the forward and backward sweep equations are

Forward sweep: $$[VLN_{abc}]_m = [A] \cdot [VLN_{abc}]_n - [B] \cdot [I_{abc}]_n$$

Backward sweep: $$[I_{abc}]_n = [c] \cdot [VLN_{abc}]_m + [d] \cdot [I_{abc}]_m$$

It was also shown that for the grounded wye–delta transformer bank, the backward sweep equation is

$$[I_{abc}]_n = [x_t] \cdot [VLN_{abc}]_n + [d] \cdot [I_{abc}]_m$$

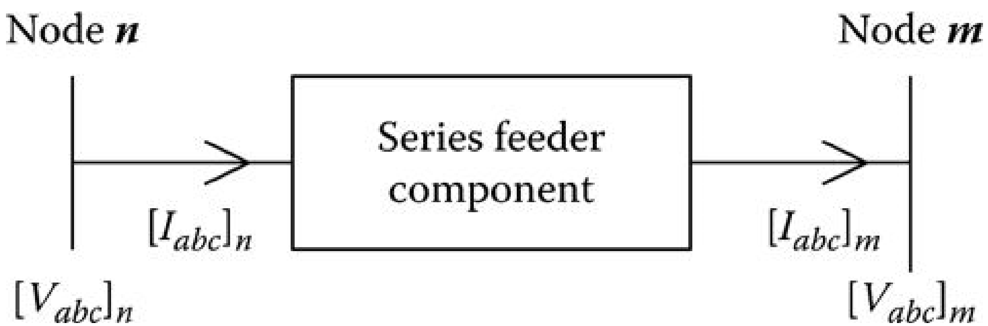


Fig.6 Standard feeder series component model [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

- In Fig. 7, nodes 4, 10, 5, and 7 are referred to as "junction nodes."
  - In the forward sweep, the voltages at all nodes down the lines from the junction nodes must be computed.
  - In the backward sweeps, the currents at the junction nodes must be summed before proceeding toward the source.
  - In developing a program to apply the modified ladder method, it is necessary for the ordering of the lines and nodes to be such that all node voltages in the forward sweep are computed and all currents in the backward sweep are computed.

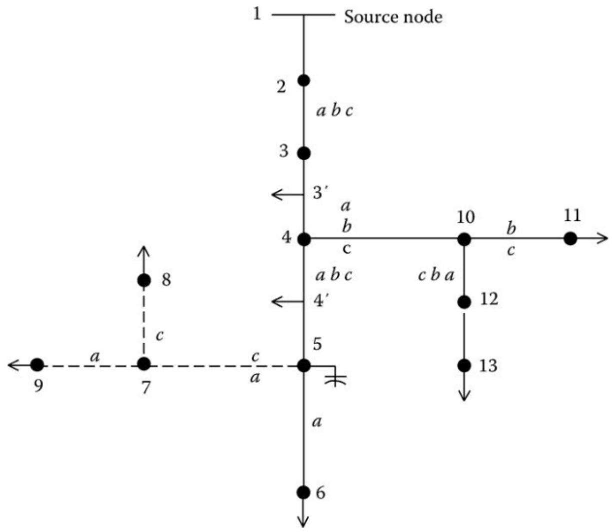
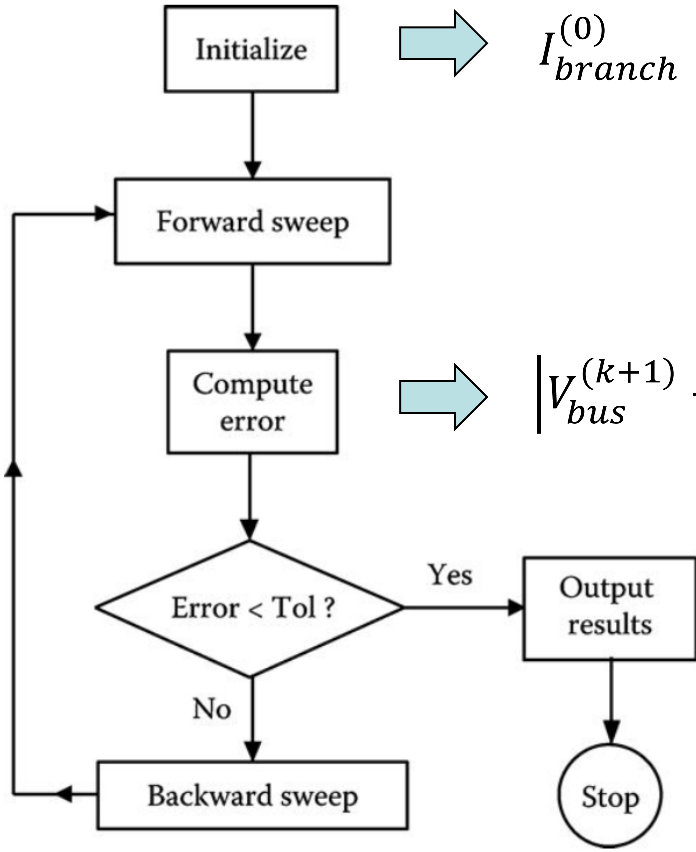


Fig.7 Typical distribution feeder [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

Forward/Backward Sweep-based Algorithm

A simple flowchart of the Forward/Backward sweep-based algorithm is shown in Fig.8.



$$I_{branch}^{(0)} = 0 \qquad V_{bus}^{(0)} = 1$$

$$\left| V_{bus}^{(k+1)} - V_{bus}^{(k)} \right|$$

Fig.8 Simple modified ladder flowchart [4]

[4] Kersting, William H. *Distribution system modeling and analysis 4th edition*. CRC press, 2017.

BIBC matrix and BCBV matrix

There are two **matrices** can be used to improve computational efficiency, which takes advantages of the topological characteristics of distribution systems and solves the distribution load flow [5]:

Bus Injection to Bus Current (BIBC) matrix: relationship between the bus current injections and branch currents

Branch current to Bus Voltage (BCBV) matrix: relationship between the branch currents and bus voltages

The reason why the BIBC and BCBV are applied:

- In conventional forward/backward sweep method, the bus voltages and line currents are calculated segments by segments (with topological characteristics) in each iteration.
- While by using the BIBC and BCBV, the two matrices are calculated only once and they have already included all topological information. BIBC/BCBV will not be updated in each iteration. Only voltage drop and branch currents will be updated.

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

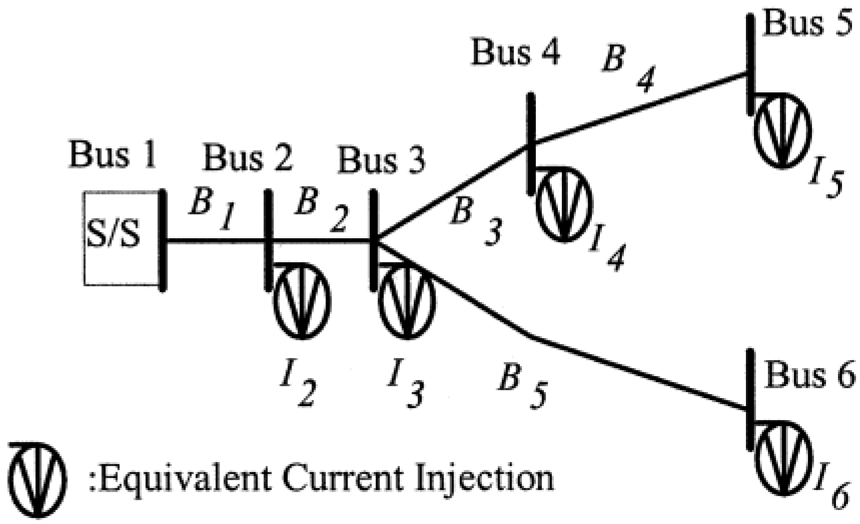


Fig.9 Equivalent Current Injection based Model of Distribution Network [5]

B is branch current
I is bus current injection

By using the KCL

$$B_1 = I_2+I_3+I_4+I_5+I_6$$

$$B_3 = I_4+I_5$$

$$B_5 = I_6$$

Relationship between the bus current injections and branch currents

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \end{bmatrix}.$$

$$[B] = [BIBC][I]$$

The constant BIBC matrix is an upper triangular matrix and contains values of 0 and 1 only.

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

To build BIBC matrix:

Step.1 For a distribution system with ***m***-branch section and ***n***-bus, the dimension of the BIBC matrix is $m \times (n-1)$.

↓

Step.2 If a line section is located between bus ***i*** and bus ***j***, copy the column of the ***i***-th bus of the BIBC matrix to the column of the ***j***-th bus and fill a +1 to the position of the ***k***-th row and the ***j***-th bus column

↓

Step.3 Repeat ***Step.2*** until all line sections are included in the BIBC matrix

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

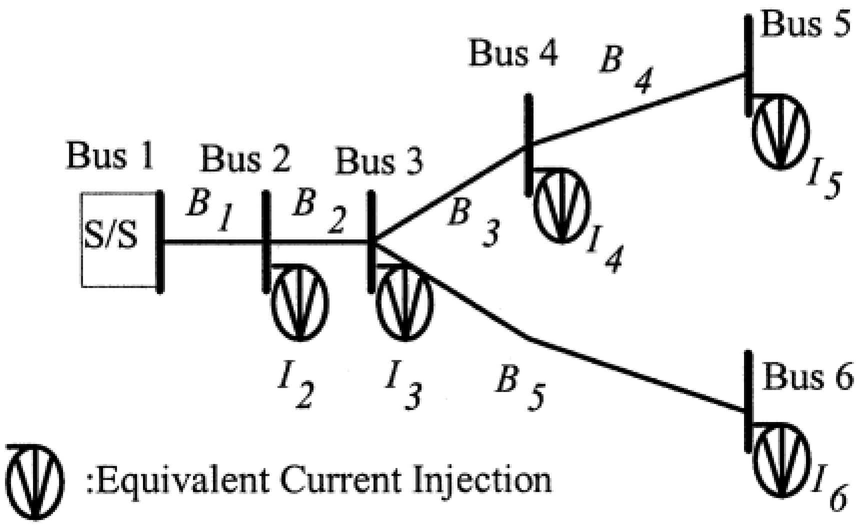


Fig.10 Equivalent Current Injection based Model of Distribution Network [5]

By using the KVL

$$V_2 = V_1 - B_1 Z_{12}$$

$$V_3 = V_2 - B_2 Z_{23}$$

$$V_4 = V_3 - B_3 Z_{34}$$

$$V_4 = V_1 - B_1 Z_{12} - B_2 Z_{23} - B_3 Z_{34}$$

Relationship between branch currents and bus voltages

$$\begin{bmatrix} V_1 \\ V_1 \\ V_1 \\ V_1 \\ V_1 \end{bmatrix} - \begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix} = \begin{bmatrix} Z_{12} & 0 & 0 & 0 & 0 \\ Z_{12} & Z_{23} & 0 & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & Z_{45} & 0 \\ Z_{12} & Z_{23} & 0 & 0 & Z_{36} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \end{bmatrix}.$$

V is bus voltage
Z is line impedance

$$[\Delta V] = [BCBV][B]$$

The constant BIBC matrix is a lower triangular matrix and contains values of 0 and line impedance only.

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

To build BCBV matrix:

Step.1 For a distribution system with ***m***-branch section and ***n***-bus, the dimension of the BCBV matrix is n × $(m-1)$.

Step.2 If a line section is located between bus ***i*** and bus ***j***, copy the column of the ***i***-th bus of the BCBV matrix to the column of the ***j***-th bus and fill the line impedance Z_{ij} to the position of the ***j***-th row and the ***k***-th bus column.

Step.3 Repeat ***Step.2*** until all line sections are included in the BCBV matrix.

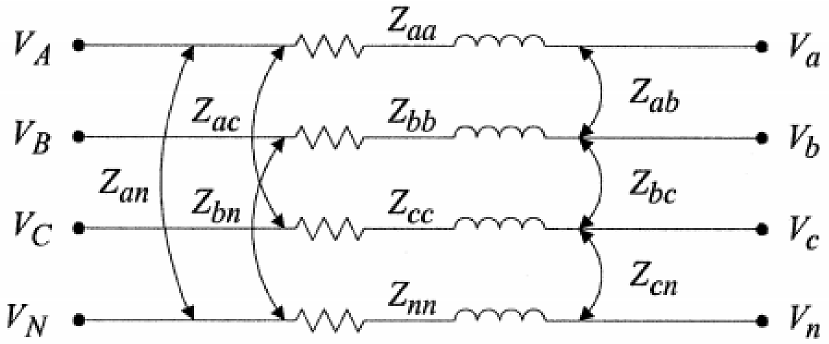
[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

Three-phase BCBV matrix

The algorithm can easily be expanded to a multiphase line section or bus.

For example, if the line section between bus ***i*** and bus ***j*** is a three-phase line section, the corresponding branch current B_i will be a 3×1 vector and the in the BIBC matrix will be a 3×3 identity matrix.

Similarly, if the line section between bus ***i*** and bus ***j*** is a three-phase line section, the Z_{ij} in the BCBV matrix is a 3×3 impedance matrix as follows.



$$[Z_{abc}] = \begin{bmatrix} Z_{aa-n} & Z_{ab-n} & Z_{ac-n} \\ Z_{ba-n} & Z_{bb-n} & Z_{bc-n} \\ Z_{ca-n} & Z_{cb-n} & Z_{cc-n} \end{bmatrix}$$

Fig.11 Three-phase line section model [5]

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

Distribution Load Flow (**DLF**) matrix is a multiplication matrix of BCBV and BIBC matrices.

$$[B] = [BIBC][I] \qquad [\Delta V] = [BCBV][B]$$

Combine two steps into one

$$[\Delta V] = [BCBV][B] = [BCBV][BIBC][I] = [\boldsymbol{DLF}][I]$$

The solution for distribution load flow can be updated and obtained iteratively as follows:

$$I_i^k = I_i^k V_i^k + jI_i^k V_i^k = \left(\frac{(P_i + jQ_i)}{V_i^k}\right)^*$$

$$[\Delta V_i^{k+1}] = [DLF][I_i^k]$$

$$[V_i^{k+1}] = [V^0] + [\Delta V_i^{k+1}]$$

- The voltage drop on each branch is computed using the DLF and **load currents**.
- The **node voltages** are computed by using the source bus voltage and **voltage drops**.

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

Step.1 Input the radial system topology data

Step.2 Form the BIBC matrix

$$[BIBC] = [I]/[B]$$

Step.3 Form the BCBV matrix

$$[BCBV] = [B]\ /[\Delta V]$$

Step.4 Calculate DLF matrix and set iteration k=0

$$[\boldsymbol{DLF}] = [BCBV][BIBC]$$

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

Step.4 Calculate DLF matrix and set iteration k=0

$$[\boldsymbol{DLF}] = [BCBV][BIBC]$$

Step.5 Update voltage and iteration k=k+1

$$I_i^k = I_i^k V_i^k + jI_i^k V_i^k = \left(\frac{(P_i + jQ_i)}{V_i^k}\right)^*$$

$$\left[\Delta V_i^{k+1}\right] = [DLF]\left[I_i^k\right]$$

$$\left[V_i^{k+1}\right] = [V^0] + \left[\Delta V_i^{k+1}\right]$$

Step.6 If $\left(\left|V_i^{k+1}\right| - \left|V_i^k\right|\right) <$ tolerance, go to next step; else, go back to ***Step. 5***

Step.7 Calculate line flows and power losses using final voltage

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

Some distribution feeders serve high-density load areas and contain loops. The proposed method introduced before can be extended for "weakly-meshed" distribution feeders.

Modification for BIBC matrix:

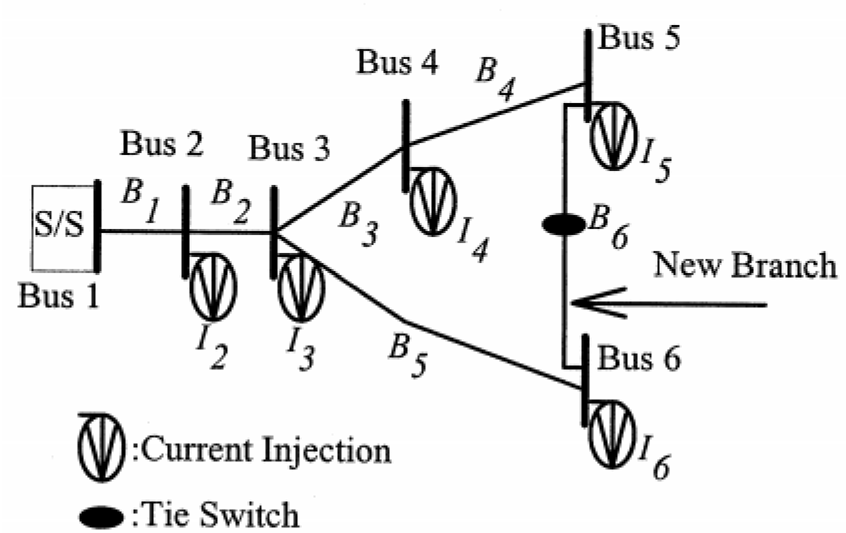


Fig.12 Simple distribution system with one loop [5]

Taking the new branch current into account, the current injections of bus 5 and bus 6 will be:

$$I_5' = I_5 + B_6$$
$$I_6' = I_6 - B_6$$

$$\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ B_6 \end{bmatrix}.$$

$$\begin{bmatrix} B \\ B_{new} \end{bmatrix} = [BIBC] \begin{bmatrix} I \\ B_{new} \end{bmatrix}$$

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

Modification for BCBV matrix:

Considering the loop shown in Fig. 12, KVL for this loop can be written as:

$$B_3 Z_{34} + B_4 Z_{45} + B_6 Z_{56} - B_5 Z_{36} = 0$$

The new BCBV matrix is:

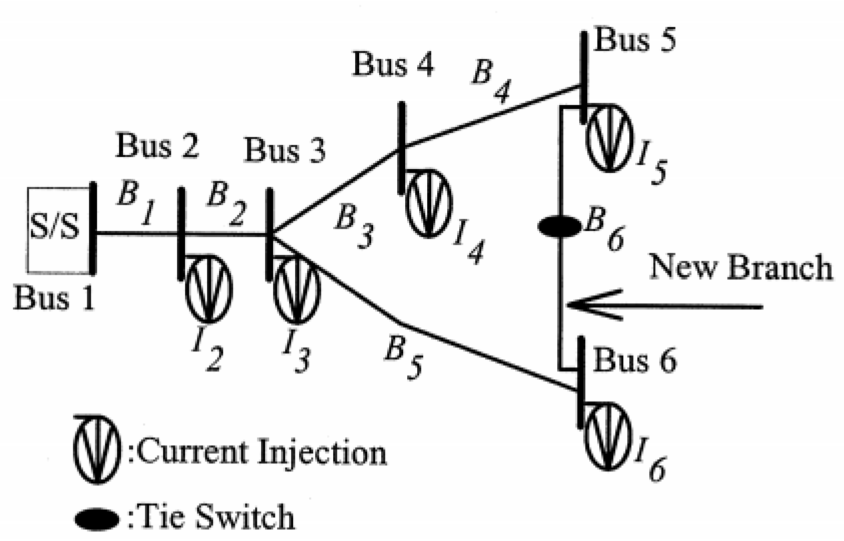


Fig.12 Simple distribution system with one loop [5]

$$\begin{bmatrix} V_1 \\ V_1 \\ V_1 \\ V_1 \\ V_1 \\ 0 \end{bmatrix} - \begin{bmatrix} V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{12} & 0 & 0 & 0 & 0 & 0 \\ Z_{12} & Z_{23} & 0 & 0 & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & 0 & 0 & 0 \\ Z_{12} & Z_{23} & Z_{34} & Z_{45} & 0 & 0 \\ Z_{12} & Z_{23} & 0 & 0 & Z_{36} & 0 \\ 0 & 0 & Z_{34} & Z_{45} & -Z_{36} & Z_{56} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \\ B_5 \\ B_6 \end{bmatrix}.$$

$$\begin{bmatrix} \Delta V \\ 0 \end{bmatrix} = [BCBV] \begin{bmatrix} B \\ B_{new} \end{bmatrix}$$

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

Modification for solution techniques:

$$\begin{bmatrix} \Delta V \\ 0 \end{bmatrix} = [BCBV] \begin{bmatrix} B \\ B_{new} \end{bmatrix}$$

$$\begin{bmatrix} B \\ B_{new} \end{bmatrix} = [BIBC] \begin{bmatrix} I \\ B_{new} \end{bmatrix}$$

$$\begin{bmatrix} \Delta V \\ 0 \end{bmatrix} = [BCBV][BIBC] \begin{bmatrix} I \\ B_{new} \end{bmatrix} = \begin{bmatrix} A & M^T \\ M & N \end{bmatrix} \begin{bmatrix} I \\ B_{new} \end{bmatrix}$$

The modified algorithm for weakly meshed networks can be expressed as

$$[\Delta V] = [A - M^T N^{-1} M][I] = [DLF][I]$$

Except for some modifications needed to be done for the BIBC, BCBV, and DLF matrices, the proposed solution techniques require no modification.

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

The proposed three-phase load flow algorithm was implemented on an 8-bus distribution system. Two methods are used for tests and the convergence tolerance is set at 0.001.

- Method 1: The Gauss implicit Z-matrix method
- Method 2: The proposed algorithm with BIBC and BCBV

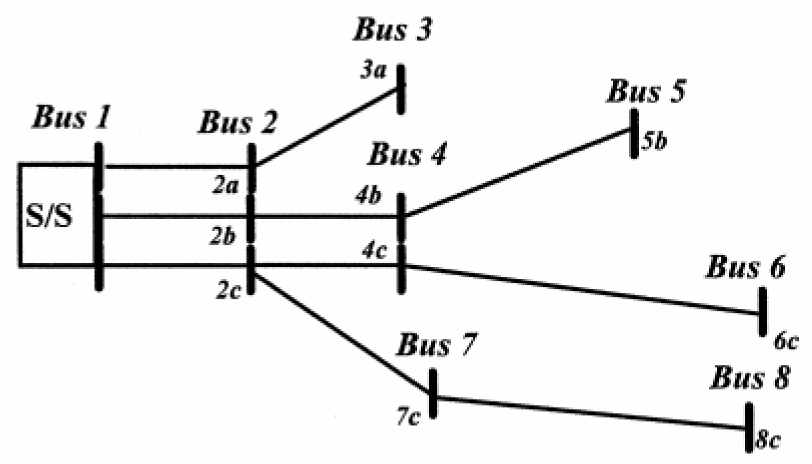


Fig.13 A 8-bus radial distribution system [5]

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

The final voltage solutions of method 1 and method 2 are shown in Tab.2.

From Tab.2, the final converged voltage solutions of method 2 are very close to the solution of method 1.

It means that the accuracy of the proposed method is almost the same as the commonly used Gauss implicit -matrix method.

Tab. 2 Final Converged Voltage Solutions [5]

Bus Number	Method 1		Method 2		Phase
	|V|(pu)	Ang. (Rad.)	|V|(pu)	Ang (Rad.).	
1	1.0000	0.0000	1.0000	0.0000	A
1	1.0000	-2.0944	1.0000	-2.0944	B
1	1.0000	2.0944	1.0000	2.0944	C
2	0.9840	0.0032	0.9839	0.0032	A
2	0.9714	-2.0902	0.9712	-2.0902	B
2	0.9699	2.0939	0.9697	2.0939	C
3	0.9833	0.0031	0.9832	0.0031	A
4	0.9653	-2.0897	0.9652	-2.0897	B
4	0.9672	2.0932	0.9669	2.0932	C
5	0.9644	-2.0898	0.9640	-2.0898	B
6	0.9652	2.0930	0.9650	2.0930	C
7	0.9686	2.0937	0.9683	2.0937	C
8	0.9674	2.0936	0.9671	2.0936	C

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

BIBC matrix and BCBV matrix

Tab.4 lists the number of iterations and the normalized execution time for both methods. It can be seen that method 2 is more efficient, especially when the network size increases,

It is because the time-consuming processes such as LU factorization and forward/backward substitution of Y-bus matrix are not necessary for method 2.

Tab. 3 Test Feeder [5]

Feeder No.	No. of Nodes	Length
1	45	1.5 km
2	90	2.5 km
3	135	3.2 km
4	180	4.0 km
5	270	7.4 km

Tab. 4 Number of iteration and Normalized Execution Time [5]

Feeder No.	Method 1		Method 2	
	NET	IT	NET	IT
1	2.6229	3	1.0000	3
2	14.426	3	2.1639	3
3	52.131	4	5.4098	4
4	131.15	4	9.0164	4
5	432.79	4	18.033	4

(1) NET means the Normalized Execution Time.
(2) IT means the Number of Iteration.
(3) Performance 1.0 is set in Method 2 for Feeder 1.

[5] Jen-Hao Teng, "A direct approach for distribution system load flow solutions," in *IEEE Transactions on Power Delivery*, vol. 18, no. 3, pp. 882-887, July 2003.

Node voltage calculations (quadratic equation)

The quadratic equation relates the voltage magnitude at the receiving end to the branch power and the voltage at the sending end.

Let us consider a distribution line model as below, the real and reactive power at the receiving end can be written as

$$P_r = \frac{V_s V_r}{Z}\cos(\theta_z - \delta_s + \delta_r) - \frac{V_r^2}{Z}\cos(\theta_z)$$

$$Q_r = \frac{V_s V_r}{Z}\sin(\theta_z - \delta_s + \delta_r) - \frac{V_r^2}{Z}\sin(\theta_z)$$

$$\cos(\theta_z - \delta_s + \delta_r) = \frac{P_r Z}{V_s V_r} + \frac{V_s V_r}{V_s}\cos(\theta_z)$$

$$\sin(\theta_z - \delta_s + \delta_r) = \frac{Q_r Z}{V_s V_r} + \frac{V_s V_r}{V_s}\sin(\theta_z)$$

$$cos^2(\theta_z - \delta_s + \delta_r) + sin^2(\theta_z - \delta_s + \delta_r) = 1$$

$$V_r^4 + 2V_r^2(P_r\mathrm{R} + Q_r\mathrm{X}) - V_s^2 V_r^2 + (P_r^2 + Q_r^2)Z^2 = 0$$

Node voltages are calculated by solving this quadratic equation

$$V_r^2 = \sqrt{V_s^2 - 2(P_r\mathrm{R} + Q_r\mathrm{X}) + \frac{(P_r^2 + Q_r^2)Z^2}{V_s^2}}$$

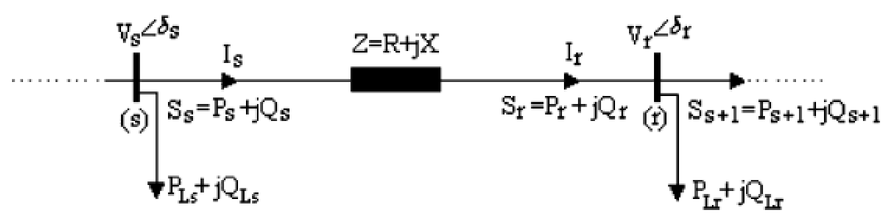


Fig.14A two-bus distribution network [3]

[3] U. Eminoglu & M. H. Hocaoglu, "Distribution Systems Forward/ Backward Sweep-based Power Flow Algorithms: A Review and Comparison Study', in *Electric Power Components and Systems*, 37:1, 91-110, 2008

Dist-Flow method (single phase)

Nodal Voltage (V_{j+1}) on Bus $j+1$: $V_{j+1}^2 = V_j^2 - 2(r_j P_j + x_j Q_j) + (r_j^2 + x_j^2)\frac{P_j^2+Q_j^2}{V_j^2}$

Active (P_{j+1}) and Reactive (Q_{j+1}) Branch Power Flow from Bus j to Bus $j+1$:

$$P_{j+1} = P_j - p_{j+1} - r_j \frac{P_j^2 + Q_j^2}{V_j^2} \qquad Q_{j+1} = Q_j - q_{j+1} - x_j \frac{P_j^2 + Q_j^2}{V_j^2}$$

$$p_{j+1} = p_{j+1}^{(c)} - p_{j+1}^{(g)} \qquad q_{j+1} = q_{j+1}^{(c)} - q_{j+1}^{(g)}$$

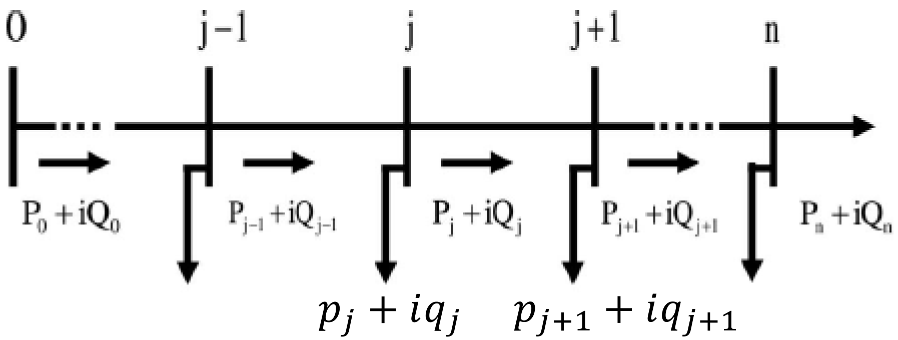


Fig.15 Dist-Flow Demonstration [6]

- $p_{j+1}^{(c)}, q_{j+1}^{(c)}$: Power consumptions at Bus j+1
- $p_{j+1}^{(g)}, q_{j+1}^{(g)}$: Power generations at Bus j+1
- r_j, x_j: Complex impedance of the line between Bus j to Bus $j+1$
- $r_j \frac{P_j^2+Q_j^2}{V_j^2}, x_j \frac{P_j^2+Q_j^2}{V_j^2}$: Active and reactive power losses of the line between Bus j to Bus $j+1$

[6] Q. Zhang, K. Dehghanpour and Z. Wang, "Distributed CVR in Unbalanced Distribution Systems With PV Penetration," in *IEEE Transactions on Smart Grid*, vol. 10, no. 5, pp. 5308-5319, Sept. 2019.

Dist-Flow method (single phase)

Forward nodal voltage calculation:

$$V_{j+1}^2 = V_j^2 - 2\left(r_j P_j + x_j Q_j\right) + \left(\mathrm{r}_j^2 + \mathrm{x}_j^2\right)\frac{\mathrm{P}_j^2 + \mathrm{Q}_j^2}{\mathrm{V}_j^2}$$

Backward branch power flow and branch power loss calculation:

$$P_{j+1} = P_j - p_{j+1} - \mathrm{r}_j \frac{\mathrm{P}_j^2 + \mathrm{Q}_j^2}{\mathrm{V}_j^2} \qquad Q_{j+1} = Q_j - q_{j+1} - \mathrm{x}_j \frac{\mathrm{P}_j^2 + \mathrm{Q}_j^2}{\mathrm{V}_j^2}$$

The calculation is ended when certain values (for example, bus voltages or the system's total active and reactive power loss mismatches) are lower than a specified error value.

Linearized Dist-Flow method (single phase)

The branch power flow and voltage constraints in Dist-Flow method have power loss terms ($\frac{\mathrm{P_i^2+Q_i^2}}{\mathrm{V_i^2}}$) that make the problem nonlinear.

There are several methods to linearize the nonlinear power loss terms:

(1) Neglect the nonlinear terms

The linearization is based on the fact that the nonlinear terms are much smaller than the linear terms. So that the nonlinear terms are neglected for the sake of developing efficient solution algorithms.

However, it is noted that such linearization neglects an accurate calculation of power loss.

$$P_{i+1} = P_i - p_{i+1} - \mathrm{r_i}\frac{\mathrm{P_i^2 + Q_i^2}}{\mathrm{V_i^2}}$$

$$Q_{i+1} = Q_i - q_{i+1} - \mathrm{x_i}\frac{\mathrm{P_i^2 + Q_i^2}}{\mathrm{V_i^2}}$$

$$V_{i+1}^2 = V_i^2 - 2(r_iP_i + x_iQ_i) + \left(\mathrm{r_i^2 + x_i^2}\right)\frac{\mathrm{P_i^2+Q_i^2}}{\mathrm{V_i^2}}$$

$$\Rightarrow$$

$$P_{i+1} = P_i - p_{i+1}$$

$$Q_{i+1} = Q_i - q_{i+1}$$

$$V_{i+1}^2 = V_i^2 - 2(r_iP_i + x_iQ_i)$$

Fully Linearized Dist-Flow method (single phase)

(2) Piecewise linear formulation (more accurate)

The quadratic calculation of branch active power losses ($P_i^{loss} = \mathrm{r_i}\frac{\mathrm{P_i^2+Q_i^2}}{\mathrm{V_i^2}}$) and reactive power losses ($Q_i^{loss} = \mathrm{x_i}\frac{\mathrm{P_i^2+Q_i^2}}{\mathrm{V_i^2}}$) can be linearized through piecewise linear formulation.

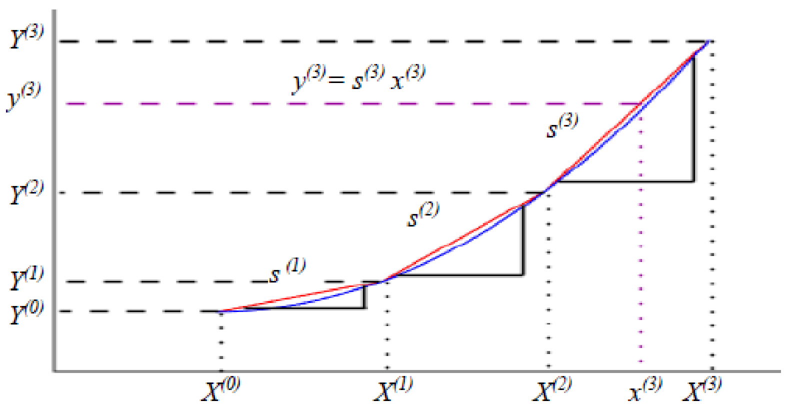


Fig.16 Piecewise Linear Formulation [7]

$$f_i(x) = \frac{R_i}{V_0^2}x^2$$

$$g_i(x) = \frac{X_i}{V_0^2}x^2$$

Linear equation slopes

$$a_{ik} = \frac{f_i\left(P_i^{(k)}\right) - f_i\left(P_i^{(k-1)}\right)}{P_i^{(k)} - P_i^{(k-1)}}$$

$$b_{il} = \frac{f_i\left(Q_i^{(l)}\right) - f_i\left(Q_i^{(l-1)}\right)}{Q_i^{(l)} - Q_i^{(l-1)}}$$

$$c_{ik} = \frac{g_i\left(P_i^{(k)}\right) - g_i\left(P_i^{(k-1)}\right)}{P_i^{(k)} - P_i^{(k-1)}}$$

$$d_{il} = \frac{g_i\left(Q_i^{(l)}\right) - g_i\left(Q_i^{(l-1)}\right)}{Q_i^{(l)} - Q_i^{(l-1)}}$$

[7] C. Zhang, Y. Xu, Z. Dong and J. Ravishankar, "Three-Stage Robust Inverter-Based Voltage/Var Control for Distribution Networks With High-Level PV," in *IEEE Transactions on Smart Grid*, vol. 10, no. 1, pp. 782-793, Jan. 2019.

Fully Linearized Dist-Flow method (single phase)

Linear calculation for the complex power loss:

$$P_i^{loss} = \sum_{k\in K_i} a_{i,k}\left(P_{i,k} - P_{i,k}^*\right) + \sum_{l\in L_i} b_{i,l}\left(Q_{i,l} - Q_{i,l}^*\right)$$

$$Q_i^{loss} = \sum_{k\in K_i} c_{i,k}\left(P_{i,k} - P_{i,k}^*\right) + \sum_{l\in L_i} d_{i,l}\left(Q_{i,l} - Q_{i,l}^*\right)$$

Piecewise power flow variable can vary only within its corresponding interval:

$$0 \le P_{i,k} \le P_i^{(k)} - P_i^{(k-1)} \qquad 0 \le Q_{i,k} \le Q_i^{(l)} - Q_i^{(l-1)}$$

$$P_i^{(k-1)} - P_i^{(k)} \le P_{i,k}^* \le 0 \qquad Q_i^{(l-1)} - Q_i^{(k)} \le Q_{i,l}^* \le 0$$

Based on the piecewise linear formulation, the fully linearized Dist-Flow with power loss is developed as:

$$P_{i+1} = P_i - p_{i+1} - \mathrm{r_i}\frac{\mathrm{P_i^2 + Q_i^2}}{\mathrm{V_i^2}} \qquad\Rightarrow\qquad P_{i+1} = P_i - p_{i+1} - P_i^{loss}$$

$$Q_{i+1} = Q_i - q_{i+1} - \mathrm{x_i}\frac{\mathrm{P_i^2 + Q_i^2}}{\mathrm{V_i^2}} \qquad\Rightarrow\qquad Q_{i+1} = Q_i - q_{i+1} - Q_i^{loss}$$

[7] C. Zhang, Y. Xu, Z. Dong and J. Ravishankar, "Three-Stage Robust Inverter-Based Voltage/Var Control for Distribution Networks With High-Level PV," in *IEEE Transactions on Smart Grid*, vol. 10, no. 1, pp. 782-793, Jan. 2019.

Extension to unbalanced three-phase systems

- Up to this point, it has only considered the single phase; however, distribution networks are inherently three-phase unbalanced.
- Also, the coupling between phases for the **system voltages requires additional approximations to simplify the unbalanced case**.

Formulations are developed by L. Gan and S. Low at Caltech (Patent number: US20150346753A1) [8]:

$$\begin{bmatrix}\left(V_j^a\right)^2\\ \left(V_j^b\right)^2\\ \left(V_j^c\right)^2\end{bmatrix}-\begin{bmatrix}(V_i^a)^2\\ \left(V_i^b\right)^2\\ (V_i^c)^2\end{bmatrix}+2\begin{bmatrix}P_{ij}^a\tilde{R}_{aa}+P_{ij}^b\tilde{R}_{ab}+P_{ij}^c\tilde{R}_{ac}+Q_{ij}^a\tilde{X}_{aa}+Q_{ij}^b\tilde{X}_{ab}+Q_{ij}^c\tilde{X}_{ac}\\ P_{ij}^a\tilde{R}_{ba}+P_{ij}^b\tilde{R}_{bb}+P_{ij}^c\tilde{R}_{bc}+Q_{ij}^a\tilde{X}_{ba}+Q_{ij}^b\tilde{X}_{bb}+Q_{ij}^c\tilde{X}_{bc}\\ P_{ij}^a\tilde{R}_{ca}+P_{ij}^b\tilde{R}_{cb}+P_{ij}^c\tilde{R}_{cc}+Q_{ij}^a\tilde{X}_{ca}+Q_{ij}^b\tilde{X}_{cb}+Q_{ij}^c\tilde{X}_{cc}\end{bmatrix}=0$$

[8] Gan, Lingwen, and Steven H. Low. "Systems and Methods for Convex Relaxations and Linear Approximations for Optimal Power Flow in Multiphase Radial Networks." U.S. Patent Application No. 14/724,757.

Extension to unbalanced three-phase systems

In single-phase distribution system, it has

$$V_j = V_i - z_{ik} \frac{P_{ij} - jQ_{ij}}{V_i^*}$$

Extend to three-phase system, it has

$$V_j^{\phi} = V_i^{\phi} - z_{ij}^{\phi} \left[S_{ij}^{\phi^*} \oslash V_i^{\phi^*} \right] \qquad S_{ij}^{\phi^*} = P_{ij}^{\phi} - jQ_{ij}^{\phi}$$

Where $V_i^{\phi} = \left[V_i^a, V_i^b, V_i^c\right]^T$, $V_j^{\phi} = \left[V_j^a, V_j^b, V_j^c\right]^T$, $P_{ij}^{\phi} = \left[P_{ij}^a, P_{ij}^b, P_{ij}^c\right]^T$, $Q_{ij}^{\phi} = \left[Q_{ij}^a, Q_{ij}^b, Q_{ij}^c\right]^T$, $z_{ij}^{\phi} \in \mathbb{C}^{3\times 3}$

$\oslash$ and $\odot$ denote the element-wise division multiplication, respectively.

- Unlike the per-phase equivalent case, multiplying by the complex conjugate of both side of the three-phase formulation will not remove the dependence on θ.
- This is due to the fact that there is a coupling between the phase at bus *i* that arises from the cross-products of the three-phase equations for the phase voltage and line current.

[9] Anmar Arif, "Distribution system outage management after extreme weather events", PhD Dissertation, Iowa State University, 2019.

Extension to unbalanced three-phase systems

To address this problem, it has observed that the voltage magnitude between the phases are similar, i.e., $|V_i^a| \approx |V_i^b| \approx |V_i^c|$, and that the phase unbalances on each bus are not very severe, so it assumes that the voltages are nearly balanced. Thus, it can approximate the phase different at bus i as:

$$\cos(\theta_i^a - \theta_i^b) = \cos\left(\frac{2}{3}\pi + \theta^*\right) = -\frac{1}{2}\cos(\theta^*) - \frac{\sqrt{3}}{2}\sin(\theta^*) \approx -\frac{1}{2}$$

$$\sin(\theta_i^a - \theta_i^b) = \sin\left(\frac{2}{3}\pi + \theta^*\right) = \frac{1}{2}\cos(\theta^*) + \frac{\sqrt{3}}{2}\sin(\theta^*) \approx \frac{\sqrt{3}}{2}$$

Where θ^* represents the relative phase unbalance, which is sufficiently small. Therefore, the nearly balanced voltages are

$$\frac{V_i^a}{V_i^b} \approx \frac{V_i^b}{V_i^c} \approx \frac{V_i^c}{V_i^a} \approx e^{j2\pi/3}$$

[9] Anmar Arif, "Distribution system outage management after extreme weather events", PhD Dissertation, Iowa State University, 2019.

Extension to unbalanced three-phase systems

It can update the **voltage magnitude** in Dist-Flow method for the unbalanced case with

$$\begin{bmatrix} |V_j^a|^2 \\ |V_j^b|^2 \\ |V_j^c|^2 \end{bmatrix} = \begin{bmatrix} |V_i^a|^2 \\ |V_i^b|^2 \\ |V_i^c|^2 \end{bmatrix} - \tilde{z}_{ij} S_{ij}^* - \tilde{z}_{ij}^* S_{ij}$$

where

$$S_{ij} = \begin{bmatrix} P_{ij}^a + jQ_{ij}^a \\ P_{ij}^b + jQ_{ij}^b \\ P_{ij}^c + jQ_{ij}^c \end{bmatrix}$$

$$\tilde{z}_{ij} = \alpha \odot z_{ij} = \begin{bmatrix} 1 & e^{-j2\pi/3} & e^{j2\pi/3} \\ e^{j2\pi/3} & 1 & e^{-j2\pi/3} \\ e^{-j2\pi/3} & e^{j2\pi/3} & 1 \end{bmatrix} \odot \begin{bmatrix} z_{ij}^{aa} & z_{ij}^{ab} & z_{ij}^{ac} \\ z_{ij}^{ba} & z_{ij}^{bb} & z_{ij}^{bc} \\ z_{ij}^{ca} & z_{ij}^{cb} & z_{ij}^{cc} \end{bmatrix}$$

α is Phase shift matrix

[9] Anmar Arif, "Distribution system outage management after extreme weather events", PhD Dissertation, Iowa State University, 2019.

Extension to unbalanced three-phase systems

Apply above equations to Dist-Flow formulation to for the extension to unbalanced three-phase

$$\begin{bmatrix} |V_j^a|^2 \\ |V_j^b|^2 \\ |V_j^c|^2 \end{bmatrix} = \begin{bmatrix} |V_i^a|^2 \\ |V_i^b|^2 \\ |V_i^c|^2 \end{bmatrix} - \begin{bmatrix} z_{ij}^{aa} + e^{-j2\pi/3} z_{ij}^{ba} + e^{j2\pi/3} z_{ij}^{ca} & z_{ij}^{ab} + e^{-j2\pi/3} z_{ij}^{bb} + e^{j2\pi/3} z_{ij}^{cb} & z_{ij}^{ac} + e^{-j2\pi/3} z_{ij}^{bc} + e^{j2\pi/3} z_{ij}^{cc} \\ e^{j2\pi/3} z_{ij}^{aa} + z_{ij}^{ba} + e^{-j2\pi/3} z_{ij}^{ca} & e^{j2\pi/3} z_{ij}^{ab} + z_{ij}^{bb} + e^{-j2\pi/3} z_{ij}^{cb} & e^{j2\pi/3} z_{ij}^{ac} + z_{ij}^{bc} + e^{-j2\pi/3} z_{ij}^{cc} \\ e^{-j2\pi/3} z_{ij}^{aa} + e^{j2\pi/3} z_{ij}^{ba} + z_{ij}^{ca} & e^{-j2\pi/3} z_{ij}^{ab} + e^{j2\pi/3} z_{ij}^{bb} + z_{ij}^{cb} & e^{-j2\pi/3} z_{ij}^{ac} + e^{j2\pi/3} z_{ij}^{bc} + z_{ij}^{cc} \end{bmatrix} \begin{bmatrix} P_{ij}^a - jQ_{ij}^a \\ P_{ij}^b - jQ_{ij}^b \\ P_{ij}^c - jQ_{ij}^c \end{bmatrix}$$

$$- \begin{bmatrix} z_{ij}^{aa^*} + z_{ij}^{ba^*}(e^{-j2\pi/3})^* + z_{ij}^{ca^*}(e^{j2\pi/3})^* & z_{ij}^{ab^*} + z_{ij}^{bb^*}(e^{-j2\pi/3})^* + z_{ij}^{cb^*}(e^{j2\pi/3})^* & z_{ij}^{ac^*} + z_{ij}^{bc^*}(e^{-j2\pi/3})^* + z_{ij}^{cc^*}(e^{j2\pi/3})^* \\ z_{ij}^{aa^*}(e^{j2\pi/3})^* + z_{ij}^{ba^*} + z_{ij}^{ca^*}(e^{-j2\pi/3})^* & z_{ij}^{ab^*}(e^{j2\pi/3})^* + z_{ij}^{bb^*} + z_{ij}^{cb^*}(e^{-j2\pi/3})^* & z_{ij}^{ac^*}(e^{j2\pi/3})^* + z_{ij}^{bc^*} + z_{ij}^{cc^*}(e^{-j2\pi/3})^* \\ z_{ij}^{aa^*}(e^{-j2\pi/3})^* + z_{ij}^{ba^*}(e^{j2\pi/3})^* + z_{ij}^{ca^*} & z_{ij}^{ab^*}(e^{-j2\pi/3})^* + z_{ij}^{bb^*}(e^{j2\pi/3})^* + z_{ij}^{cb^*} & z_{ij}^{ac^*}(e^{-j2\pi/3})^* + z_{ij}^{bc^*}(e^{j2\pi/3})^* + z_{ij}^{cc^*} \end{bmatrix} \begin{bmatrix} P_{ij}^a + jQ_{ij}^a \\ P_{ij}^b + jQ_{ij}^b \\ P_{ij}^c + jQ_{ij}^c \end{bmatrix}$$

$$P_j^{\phi} = P_i^{\phi} - p_i^{\phi} - r_{ij}^{\phi} \frac{\left(P_i^{\phi}\right)^2 + \left(Q_i^{\phi}\right)^2}{\left(V_i^{\phi}\right)^2} \qquad Q_j^{\phi} = Q_i^{\phi} - q_i^{\phi} - z_{ij}^{\phi} \frac{\left(P_i^{\phi}\right)^2 + \left(Q_i^{\phi}\right)^2}{\left(V_i^{\phi}\right)^2}$$

$$r_{ij}^{\phi} = \mathrm{real}(\alpha \odot z_{ij})$$

$$z_{ij}^{\phi} = \mathrm{imag}(\alpha \odot z_{ij})$$

[9] Anmar Arif, "Distribution system outage management after extreme weather events", PhD Dissertation, Iowa State University, 2019.

Validation

In [9], the validation of this unbalanced approximation method VS OpenDSS simulated results.

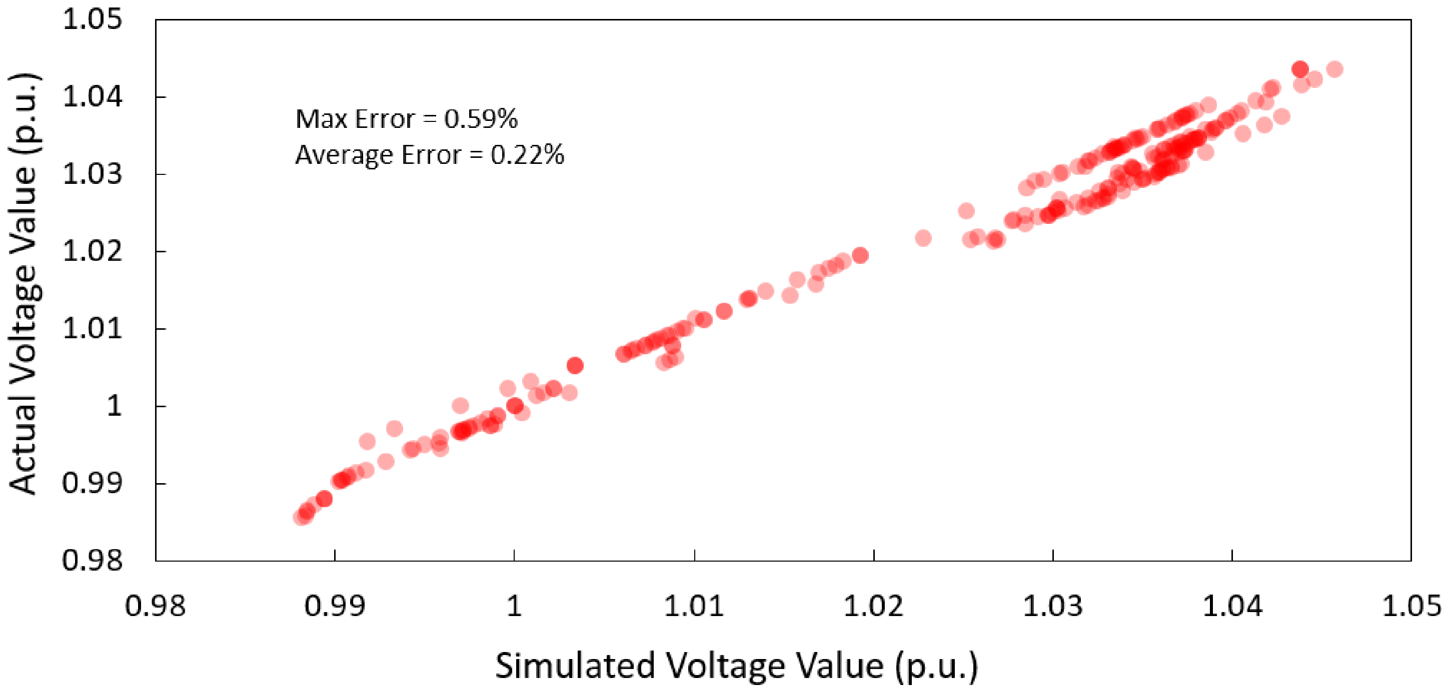


Fig.17 IEEE 123-bus system: actual vs simulated results [9]

[9] Anmar Arif, “Distribution system outage management after extreme weather events”, PhD Dissertation, Iowa State University, 2019.

Power flow in OpenDSS

However, OpenDSS manual [10] says that:

It is common in distribution power flow papers for the authors to claim that radial distribution feeders with low X/R ratios are difficult to solve. This usually refers to certain traditional Newton-Raphson formulations like those used in positive-sequence transmission system models. To the best of our knowledge, OpenDSS has never suffered from this problem and solves radial circuits quite handily.

The two basic power flow solution types are

1. Iterative power flow
2. Direct solution

For the iterative power flow, nonlinear elements such as loads and distributed generators are treated as injection sources. In the Direct solution, they are included as admittances in the system admittance matrix, which is then solved directly without iterating. Either of these two types of solutions may be used for any of the several solution modes by setting the global LoadModel property to “Admittance” or “Powerflow” (can be abbreviated A or P). The default is “Powerflow”.

There are two iterative power flow algorithms currently employed:

1. "Normal" current injection mode (default)
2. "Newton" mode.

[10] “Reference guide: The OpenDSS”, [online]: https://www.epri.com/#/pages/sa/opendss?lang=en

Modified Newton-Raphson method

In [11], A modified Newton method for radial distribution system is derived in which the Jacobian matrix is in UDU^T form, where U is a constant upper triangular matrix depending only on system topology and D is a block diagonal matrix. With this formulation, the conventional steps of forming the Jacobian matrix are replaced by back/forward sweeps on radial feeders with equivalent impedances.

In conventional Newton-Raphson method, the power flow problem is to solve

$$\begin{bmatrix} H & N \\ J & L \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta V/V \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}$$

where

$$H_{ij} = -V_i V_j \left(G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij}\right), j \neq i$$

$$H_{ii} = V_i \sum_{j\in N_i, j\neq i} V_j \left(G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij}\right)$$

$$N_{ij} = -V_i V_j \left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}\right), j \neq i$$

$$N_{ii} = -V_i \sum_{j\in N_i, j\neq i} V_j \left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}\right) - 2V_i^2\, G_{ii}$$

$$J_{ij} = V_i V_j \left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}\right), j \neq i$$

$$J_{ii} = -V_i \sum_{j\in N_i, j\neq i} V_j \left(G_{ij}\cos\theta_{ij} + B_{ij}\sin\theta_{ij}\right)$$

$$L_{ij} = -V_i V_j \left(G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij}\right), j \neq i$$

$$L_{ii} = -V_i \sum_{j\in N_i, j\neq i} V_j \left(G_{ij}\sin\theta_{ij} - B_{ij}\cos\theta_{ij}\right) + 2V_i^2\, B_{ii}$$

[11] F. Zhang and C. S. Cheng, "A Modified Newton Method for Radial Distribution System Power Flow Analysis," in *IEEE Transactions on Power System*, vol. 12, no. 1, pp. 882-887, Feb 1997.

Modified Newton-Raphson method

Assumption 1: small voltage difference between two adjacent nodes (typical distribution lines are short and line flows are not high).
Assumption 2: no shunt branches (all the shunt branches can be converted to node power injections using initial and updated node voltages).

Therefore, the Jacobian matrix can be approximated as:

$$H_{ij} \approx V_i V_j \left(B_{ij} \cos \theta_{ij}\right), j \neq i$$

$$N_{ij} \approx -V_i V_j \left(G_{ij} \cos \theta_{ij}\right), j \neq i$$

$$J_{ij} \approx V_i V_j \left(G_{ij} \cos \theta_{ij}\right), j \neq i$$

$$L_{ij} \approx V_i V_j \left(B_{ij} \cos \theta_{ij}\right), j \neq i$$

$$H_{ii} \approx -V_i \sum_{j \in N_i, j \neq i} V_j \left(B_{ij} \cos \theta_{ij}\right)$$

$$N_{ii} \approx V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \cos \theta_{ij}\right)$$

$$J_{ii} \approx -V_i \sum_{j \in N_i, j \neq i} V_j \left(G_{ij} \cos \theta_{ij}\right)$$

$$L_{ii} \approx -V_i \sum_{j \in N_i, j \neq i} V_j \left(B_{ij} \cos \theta_{ij}\right)$$

The matrices H, N, J and L all have the same properties (symmetry, sparsity pattern) as the Nodal Admittance Matrix.

[11] F. Zhang and C. S. Cheng, “A Modified Newton Method for Radial Distribution System Power Flow Analysis," in *IEEE Transactions on Power System*, vol. 12, no. 1, pp. 882-887, Feb 1997.

Modified Newton-Raphson method

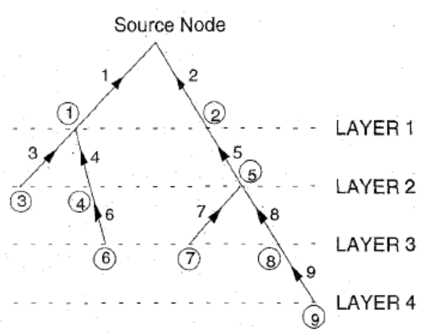
Hence, the matrices H, N, J and L can be formed as:

$$H = L = A_{n-1} D_B A_{n-1}^T$$
$$J = -N = A_{n-1} D_G A_{n-1}^T$$

where D_B and D_B are diagonal matrices with diagonal entries to be $V_i V_j B_{ij} \cos\theta_{ij}$ and $V_i V_j G_{ij} \cos\theta_{ij}$, respectively. Therefore, the conventional Newton Raphson can be rewritten as:

$$\begin{bmatrix} A_{n-1} & 0 \\ 0 & A_{n-1} \end{bmatrix} \begin{bmatrix} D_B & -D_G \\ D_G & D_B \end{bmatrix} \begin{bmatrix} A_{n-1}^T & 0 \\ 0 & A_{n-1}^T \end{bmatrix} \begin{bmatrix} \Delta\theta \\ \Delta V/V \end{bmatrix} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}$$

A_{n-1} is an upper triangular matrix with all diagonal entries to be 1 and all non-zero off-diagonal entries to be -1.



$$A_{n-1} = \begin{bmatrix} 1 & & -1 & -1 & & & & & \\ & 1 & & & -1 & & & & \\ & & 1 & & & & & & \\ & & & 1 & & -1 & & & \\ & & & & 1 & & -1 & -1 & \\ & & & & & 1 & & & \\ & & & & & & 1 & & \\ & & & & & & & 1 & -1 \\ & & & & & & & & 1 \end{bmatrix}$$

[11] F. Zhang and C. S. Cheng, “A Modified Newton Method for Radial Distribution System Power Flow Analysis," in *IEEE Transactions on Power System*, vol. 12, no. 1, pp. 882-887, Feb 1997.

Modified Newton-Raphson method

By now it has shown that the Jacobian matrix can be formed as the product of three square matrices. It can be solved by back/forward sweeps as well. It defines:

$$\dot{E} = \Delta\theta + j\,\Delta V/V$$

$$\dot{S} = \Delta P + j\Delta Q$$

$$\dot{W} = D_B + jD_G$$

Therefore, the formulations in Newton-Raphson method can be modified:

$$A_{n-1}\dot{W}A_{n-1}^{T}\dot{E}=\dot{S}_L \quad \Rightarrow \quad \begin{cases} A_{n-1}\dot{S}_L = \dot{S} \\ \dot{W}A_{n-1}^{T}\dot{E}=\dot{S}_L \end{cases}$$

When solving $\dot{E}$, the diagonal matrix $\dot{W}$ can be inverted for each line. The diagonal in $\dot{W}^{-1}$ is denoted as the equivalent line impedance:

$$Z_{eq}=R_{eq}+jX_{eq}$$

[11] F. Zhang and C. S. Cheng, "A Modified Newton Method for Radial Distribution System Power Flow Analysis," in *IEEE Transactions on Power System*, vol. 12, no. 1, pp. 882-887, Feb 1997.

Modified Newton-Raphson method

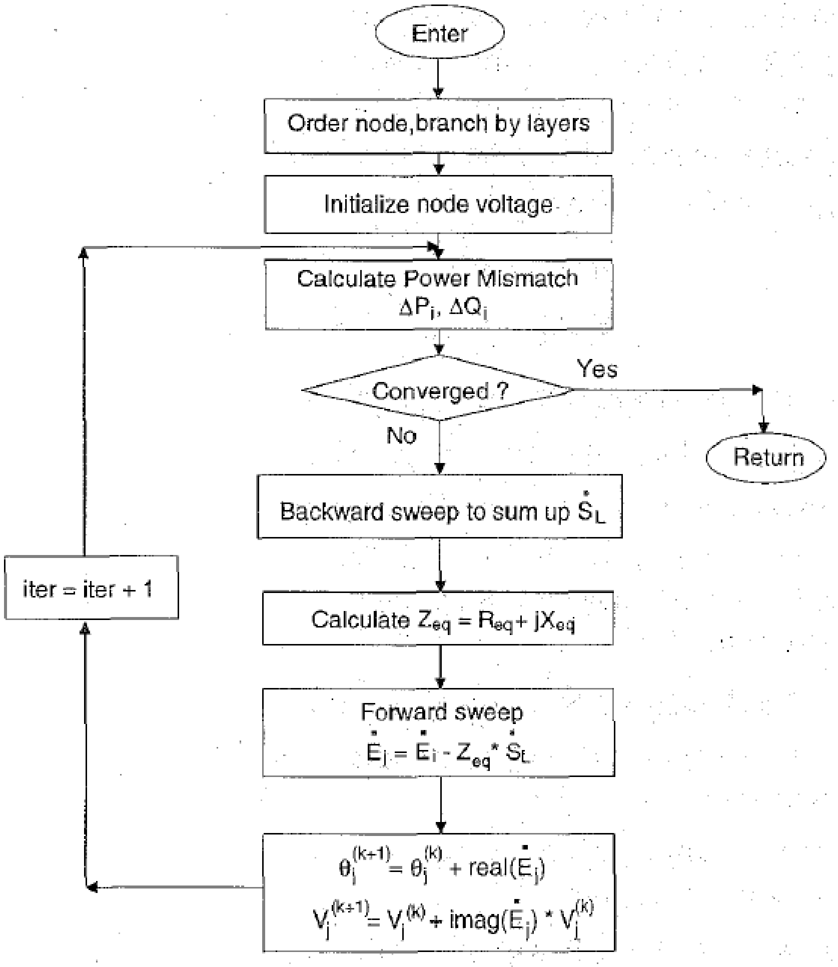


Fig.18 Flowchart of the modified Newton-Raphson method [11]

$A_{n-1}\dot{S}_L = \dot{S}$ Backward sweep

$W A_{n-1}^T \dot{E} = \dot{S}_L$ Forward sweep

[11] F. Zhang and C. S. Cheng, "A Modified Newton Method for Radial Distribution System Power Flow Analysis," in *IEEE Transactions on Power System*, vol. 12, no. 1, pp. 882-887, Feb 1997.

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[9] Anmar Arif, "Distribution system outage management after extreme weather events", PhD Dissertation, Iowa State University, 2019.

[10] "Reference guide: The OpenDSS", [online]: https://www.epri.com/#/pages/sa/opendss?lang=en

[11] F. Zhang and C. S. Cheng, "A modified newton method for radial distribution system power flow analysis," in *IEEE Transactions on Power System*, vol. 12, no. 1, pp. 882-887, Feb 1997.

Thank you!